Eigenvalue Correlations for Banded Matrices

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Abstract

We study the evolution of the distribution of eigenvalues of a $N \times N$ matrix ensemble subject to a change of variances of its matrix elements. Our results indicate that the evolution of the probability density is governed by a Fokker-Planck equation similar to the one governing the time-evolution of the particle-distribution in Wigner-Dyson gas, with relative variances now playing the role of time. This is also similar to the Fokker-Planck equation for the distribution of eigenvalues of a $N \times N$ matrix subject to a random perturbation taken from the standard Gaussian ensembles with perturbation strength as the "time" variable. This equivalence alongwith the already known correlations of standard Gaussian ensembles can therefore help us to obtain the correlations for various physically-significant cases modeled by random banded Gaussian ensembles.

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It is now well-established that the statistics of energy levels of complex quantum systems e.g. chaotic systems, disordered systems can be well-modeled by the ensembles of large random matrices [1]. The nature of underlying quantum dynamics of these systems divides the applicable random matrix ensembles (RME) into two major classes. The ensembles of the full random matrices (RFME) [2] have been very successful in modelling the statistical properties of systems with delocalized quantum dynamics. On the other hand, presence of the localized quantum dynamics requires a consideration of the Band random matrices ensembles (RBME) which can generally be described as a $N \times N$ matrix with non-zero elements effectively within a band of the width $b > 1$ around the main diagonal. The presence of these two major classes can be understood as follows. In the matrix representation of an operator, an off-diagonal matrix element describes the overlapping of the eigenfunctions in the basis space. For the delocalized quantum dynamics, the eigenfunctions are extended and the overlapping between them and therefore off-diagonal matrix elements are of the same order. The ensembles of such matrices are referred here as the RFME. In contrast, the localization of the eigenfunctions implies a decaying tendency of overlapping between them, leading to a diminishing strength of the corresponding off-diagonal matrix elements and therefore RBM. Our aim in this paper is to suggest a general method for the statistical studies of RBME.

Various studies of the eigenfunctions of complex systems have revealed the existence of various types of localizations, intermediate between fully localized eigenfunctions (occurs only when system size $L \to \infty$) and fully extended eigenstates (implying that the localization length $\geq$ the system size). For example, the eigenfunctions, in a weakly disordered potential exhibiting the Anderson metal-insulator transition, are essentially structureless and extended and the statistics here can be well-modeled by FRME. In the insulator regime, a complete localization of eigenfunctions leads to a zero correlation among them and a Poisson statistics for energy levels. However the eigenfunctions in the critical region near the Anderson transition reveal a special feature "multifractality" in their structure [3]. The mul-

tifractal nature of a wavefunction is commonly accepted to be related to its space structure which also reveals itself in their overlapping with each other; the two fractal wavefunctions, however sparse they are, overlap strongly in contrast to two fully localized states. The nature of the energy level statistics being related to the structure of eigenfunctions as well as their overlapping, the latter's special feature should manifest also in the former. Indeed the presence of multifractality leads to a new type of universal critical level statistics which is described by a set of critical exponents and is different from both Wigner-Dyson (FRME) as well as Poisson statistics [4,5]. Recently there have been suggestions regarding the existence of such eigenfunctions in wide variety of physical systems. For example, a Coulomb impurity inside an integrable square billiard leads to multifractal eigenfunctions in momentum representation however small is the strength of the potential [6]. It also seems quite possible that the multifractality is a quantum mechanical manifestation of the special nature of underlying classical dynamics, intermediate between integrability (fully localized eigenfunctions) and chaos (fully extended eigenfunctions) [3].

Recent research on complex systems has indicated that RBME, with zero mean value of all the matrix elements and the variance $< |H_{ij}|^2 > \propto a(|i-j|)$, can serve as good models for systems with multifractal eigenfunctions [7,6,8]; here the function $a(r)$ decays with r either as a power law or exponentially (or faster) at $r >> 1$. However various other RBM type structures appear in many other physical contexts, for example, nuclei, atoms [10], solid state [11], quantum chaos [12]. One such example is that of WRBM (W stands for Wigner) [9] where the mean value of diagonal elements increases linearly along the main diagonal ($< H_{nn} >= \alpha n$). These matrices have been shown to be a good model, for example, for the tight-binding Hamiltonian of a quantum particle in a 1-D disordered system subject to a constant electric field as well as for heavy atoms and nuclei [11]. Another example is of the ensemble of banded matrices with diagonal elements fluctuating much stronger than the off-diagonal ones ($< |H_{ii}|^2 > / < |H_{ij}|^2 > \propto b >> 1$), also known as RBM with preferential basis. A very interesting problem of two interacting particles propagating in a quenched random potential can effectively be mapped on to this class [14].

The nature of a localized dynamics leaves its imprints on the distribution properties of the matrix elements of the generator of motion e.g. Hamiltonian. It is therefore desirable to study RBME with various distributions and recently a lot of effort has been applied in this direction. However the techniques suggested so far [7,11,13] depend strongly on the type of distribution chosen and mostly give approximate results. Our aim in this paper is to suggest a method which leads to results in a generic form vaild for most of the RBM models. This we achieve by mapping the problem to the study of a particular class of transition ensembles, the one arising during the Poisson $\rightarrow$ standard Gaussian transitions under a perturbation taken from the standard Gaussian ensembles (SGE). The pre-existing information regarding the correlations in the latter case therefore can help us to obtain the same for RBME.

We proceed as follows. Our interest is in the evolution of the eigenvalue distribution of H, taken from an ensemble of hermitian matrices, due to variation of the variances of its matrix elements. We choose the distribution $\rho(H)$ of matrix H to be a Gaussian, $\rho(H,y) = C\exp(-\sum_{k\leq l}\alpha_{kl}H_{kl}^2)$ with C as the normalization constant and y as the matrix of relative variances $y_{kl} = \frac{\alpha_{kl}g_{kl}}{\alpha_{kk}g_{kk}}$. Let $P(\{\mu_i\},y|H_0)$ be the probability of finding eigenvalues λ_i of H between μ_i and $\mu_i + \mathrm{d}\mu_i$ at a given y (with H_0 as an initial condition),

$$P(\{\mu_i\},y|H_0) = \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i)\rho(H,y)\mathrm{d}H \tag{1}$$

As the α-dependence of P in eq.(1) enters only through $\rho(H)$ ($\frac{\partial\rho(H)}{\partial\alpha_{kl}} = [(2\alpha_{kl})^{-1} - H_{kl}^2]\,\rho(H)$), this equality followed by a repeated use of the partial integration alongwith eigenvalue equation $H = O^T\Lambda O$, with Λ as the eigenvalue matrix and O an orthogonal matrix, leads to following,

$$2\frac{\partial P(\{\mu_i\},y)}{\partial\alpha_{kl}} = \frac{1}{\alpha_{kl}^2}I_{kl} \tag{2}$$

where

$$I_{kl} = \sum_n \frac{\partial}{\partial\mu_n}\int \mathrm{d}H\rho(H)\frac{\partial}{\partial H_{kl}}\left(\prod_i \delta(\mu_i - \lambda_i)O_{nk}O_{nl}\right) \tag{3}$$

Let us first study the case with same variance for all the diagonal matrix elements such that $g_{kl}\alpha_{kl} = \alpha$ for $k = l$ ($g_{kl} = 1 + \delta_{kl}$) while keeping the variances ($=\alpha_{kl}^{-1}$) of the off-

diagonals arbitrary. By expressing $\sum_{k<l}\left(\frac{1}{\alpha}-\frac{1}{g_{kl}\alpha_{kl}}\right)...=\frac{1}{\alpha}\sum_{k\leq l}...-\sum_{k\leq l}\frac{1}{g_{kl}\alpha_{kl}}..$ and using eq.(2), we obtain the following relation

$$2\sum_{k<l}(1-y_{kl})\,y_{kl}\frac{\partial P}{\partial y_{kl}}=\sum_{k\leq l}\frac{1}{g_{kl}\alpha_{kl}}I_{kl}-\frac{1}{\alpha}\sum_{k\leq l}I_{kl} \tag{4}$$

The left hand side of above equation, summing only over all distinct α_{kl}, can be rewritten as $\frac{\partial P}{\partial Y}$ with Y given by the condition that $\frac{\partial}{\partial Y}=2\sum_{k<l}y_{kl}(1-y_{kl})\frac{\partial}{\partial y_{kl}}$. The first term on the right hand side (eq.(4)) can further be simplified by first using a partial integration and subsequently the eigenvalue equation for H: $\sum_{k\leq l}\frac{1}{\alpha_{kl}}I_{kl}=\sum_n\frac{\partial}{\partial\mu_n}(\mu_n P)$. The second term can similarly be reduced by differentiating the terms inside the brackets in (3) giving us two integrals. The use of orthogonality relation of matrix O in the first integral so obtained and the equality $\sum_{k\leq l}\frac{\partial O_{nk}O_{nl}}{\partial H_{kl}}=\sum_m\frac{1}{\lambda_m-\lambda_n}$ in the second [15] leads to a F-P equation

$$\begin{aligned}\frac{\partial P}{\partial Y}&=\sum_n\frac{\partial}{\partial\mu_n}(\mu_n P)\\&\quad+\frac{1}{\alpha}\sum_n\frac{\partial}{\partial\mu_n}\left[\frac{\partial}{\partial\mu_n}+\sum_{m\neq n}\frac{\beta}{\mu_n-\mu_m}\right]P\end{aligned} \tag{5}$$

where $\beta=1$. By using the unitarity of eigenvectors and following the same steps, it can be proved for complex Hermitian case too (now $\beta=2$). It should be noted here, for the later use, that the required form of eq.(5) is obtained by applying an appropriate partitioning of sums appearing in eq.(4) which leads to rewriting of the terms containing all the unequal coefficients α_{kl} as the derivative with respect to parameter Y. The coefficient appearing with drift and diffusion terms in eq.(5) is the one common to many of the matrix elements.

The definition of Y depends on the relative value of variances of the off-diagonals as discussed below for some cases:

Case I. When all the off-diagonals have same variances such that $\alpha_{kl}=\alpha'$ (for $k\neq l$)

In this case, as $y=\frac{\alpha'}{\alpha}$, therefore $Y=2\log\frac{y}{|y-1|}$.

Case II. When variances of the off-diagonals change with respect to distance from the diagonal such that $\alpha_{kl}=\alpha_r$ (for $r=|k-l|>b>0$) and $\alpha_{kl}=\alpha$ for $r\leq b$.

Now all the off-diagonals (with $y_r\neq 1$) contribute separately to Y, $Y=2\sum_{r=b+1}^{N}\log\frac{y_r}{|y_r-1|}$ where $y_r=\frac{\alpha_r}{\alpha}$.

For all $y_r > 1$, this represents a RBM with strongly fluctuating diagonal elements (known as RBM with preferential basis). The case with all $y_r < 1$ describes the standard RBM, used to model, for example, the spectral statistics of the tight binding Hamiltonian of a quantum particle in a 1-D system with long range random hoppings [7] and also for quasi 1-D disordered wires [16].

Case III. When the off-diagonals form various groups with different varainces (with same variance for each matrix element in one such group).

Now contribution to Y comes from each such group. For example, with M groups of variances given by α_g^{-1} ($g = 1 \to M$, $\alpha_g \neq 1$), $Y = 2\sum_{g=1}^{M} \log\frac{y_g}{|y_g-1|}$ where $y_g = \frac{\alpha_g}{\alpha}$. This will be helpful to model the more general cases where the variance $< |V_{ij}|^2 >$ is dependent on both indices i, j instead of their difference $|i-j|$.

The steady state of eq.(5), $P(\{\mu_i\}, \infty) \equiv P_\infty = \prod_{i<j} |\mu_i - \mu_j|^\beta \mathrm{e}^{-\frac{\alpha}{2}\sum_k \mu_k^2}$, is achieved for $Y \to \infty$ which corresponds to $y \to 1$ in case (I), $y_r \to 1$ in case (II) and $y_{kl} \to 1$ for the case (III). This indicates that, in the steady state limit, system tends to belong to the standard Gaussian ensembles. Eq.(5) (later referred as variance-variation or VV case) is formally the same as the F-P equation governing the Brownian motion of particles in Wigner-Dyon gas [2] with transition parameter being the relative variance in the former and time in the latter. This is also similar to the F-P equation for the eigenvalue distribution of a hermitain matrix $H = H_0 + \tau V$ undergoing random perturbation V of strength τ and taken from a SGE with arbitrary initial condition H_0 (later referred as perturbation variation or PV case) [17]; here τ acts as the transition parameter. $P(\{\mu_i\}, Y)$ can therefore be obtained by the same procedure as used in PV case which is briefly given as follows. The transformation $\Psi = P/\sqrt{P_\infty}$ reduces eq.(5) to a "Schrodinger equation" form, $\frac{\partial\Psi}{\partial Y} = \hat{H}\Psi$, where the 'Hamiltonian' $\hat{H}$ turns out to be a Calogeo-Moser (CM) Hamiltonian [19,20], $\hat{H} = \sum_i \frac{\partial^2}{\partial\mu_i^2} - \frac{1}{4}\sum_{i<j} \frac{\beta(\beta-2)}{(\mu_i-\mu_j)^2} - \frac{\alpha^2}{4}\sum_i \mu_i^2$, and has well-defined eigenstates and eigenvalues [21]. The "state" Ψ or $P(\{\mu_i\}, Y|H_0)$ can therefore be expressed as a sum over its eigenvalues and eigenfunctions which on integration over all the initial conditions H_0 leads to the joint probability distribution $P(\{\mu_i\}, Y)$ and thereby correlations. Unfortunately, due to technical

problems, the latter could be evaluated only for the transitions with final steady state (limit $\Lambda \to \infty$) as GUE [20]. However a set of hierarchic relations among the correlators for all transitions in PV case, and therefore for VV case, can be obtained by a direct integration of the F-P equation for P (eq.(5)) [19,20]. For example, in large N-limit, the evolution of level density $\rho(\mu, Y)$ is governed by the Dyson-Pastur equation [19,20] which results in a semi-circular form for ρ (thus agreeing well with the ρ obtained in [18] by super-symmetry technique). The relation $\rho(\mu, Y) = N^{-1} \sum_n \rho(\mu, n, Y)$ can further be used to obtain the equation for the local density of states (LDOS) $\rho(\mu, n, Y)$, a more informative and experimentally accessible quantity (referred below as ρ_n),

$$\frac{\partial \rho_n}{\partial Y} = -\beta \frac{\partial}{\partial \mu} \left(\sum_m \mathbf{P} \int \mathrm{d}\mu' \frac{\rho_m}{\mu - \mu'} \right) \rho_n. \tag{6}$$

This nonlinear equation gives uniquely $\rho(E, n, Y)$ starting from an initial $\rho(E, n, 0)$.

For localization studies, it is appropriate to choose the initial ensemble H_0 as that of diagonal matrices ($P(H_0) \propto \mathrm{e}^{-\frac{\alpha}{2} \sum_i H_{ii}^2}$ with $V_{ii} = \mu_{0i}$ and all $y \to \infty$) which corresponds to Poisson distribution for eigenvalues and $Y = 0$. With equilibrium distribution ($Y \to \infty$) given by SGE, this case thus represents a Poisson $\to$ SG transition with Y as a transition parameter and the intermediate ensembles representing various RBMEs depending on the type of y_r's. Thus the correlations here will be similar to those in the Poisson$\to$SG transition in SGE (PV case with H_0 taken from a Poisson ensemble). We already know that the transition for PV case is abrupt for large dimensions and finite τ and a rescaling of τ by mean spacing $D^2 (\propto N)$ is required to make it smooth, the new transition parameter being $\Lambda = \frac{\tau}{D^2}$. A similar rescaling should also be applied to Y in VV case for the same reason; here too $D^2 \propto N$ [18]. (This N-dependence of D also follows from analogy of the evolution-equations for level-densities in the two cases).

Fortunately the two-point correlation $R_2(r; \Lambda)$ for Poisson $\to$ GUE transition in PV case has already been obtained [20] and, as discussed above, is also valid for the VV case (now $\Lambda = Y/D^2$) .

$$R_2(r;\Lambda) - R_2(r;\infty) = \frac{4}{\pi}\int_0^\infty \mathrm{d}x \int_{-1}^1 \mathrm{d}z\, \cos(2\pi r x)\, \exp\left[-8\pi^2\Lambda x(1+x+2z\sqrt{x})\right] \left(\frac{\sqrt{(1-z^2)}(1+2z\sqrt{x})}{1+x+2z\sqrt{x}}\right)$$

where $R_2(r,\infty) = 1 - \frac{\sin^2(\pi r)}{\pi^2 r^2}$ (the GUE limit). As can easily be checked, above equation has the correct limiting behaviour, that is, $R_2 = 0$ for $\Lambda \to 0$ (the Poisson case $Y = 0$) and $R_2 = R_2(r;\infty)$ for $\Lambda \to \infty$. As obvious from eq.(7), R_2 for intermediate ensembles will depend on definition of Y and therefore on the nature of localization which results in various types of level-statistics. For example, let us calculate the correlation for one such case, namely, $H_{ij} = G_{ij}a(|i-j|)$ with G a typical member of SGE and $a(r) = 1$ and $(b/r)^\sigma$ for $r \leq b$ and $> b$ ($b >> 1$) respectively. For this case, $y_r = (\frac{r}{b})^\sigma$ and $Y \approx 2\sum_{r=b+1}^N \left(\frac{b}{r}\right)^{2\sigma}$ which gives $\Lambda \propto N^{2-2\sigma}$. For $\sigma > 1$ with $N \to \infty$, therefore, the eigenvalue statistics approches Poisson limit, Λ being very small. Similarly for $\sigma < 1$, Λ is sufficiently large and the eigenvalue statistics approches SG limit. For $\sigma = 1$, the independence of Λ from N leads to an eigenvalue statistics, very different from that of SGE or Poisson and therefore agrees well with the results obtained in [7] by using non-linear σ-model technique. Further R_2, (eq.(7)) is also in good accordance with the one given in [7] (which can be seen by a direct substitution of eq.(52) of [7] in eq.(17) of [20]).

It is possible to have physical situations when the matrix elements form various groups such that those in one group have the same value for $\alpha_{kl}g_{kl} = \alpha_r$ ($r = 1 \to M$ with M as total number of groups). This case can similarly be treated by using $2\sum_{r=1}^M \left(\frac{1}{\alpha_s} - \frac{1}{\alpha_r}\right)\alpha_r^2 \frac{\partial P}{\partial \alpha_r} = \frac{1}{\alpha_s}\sum_r I_r - \sum_r \frac{1}{\alpha_r} I_r$ with $I_r = \sum_{\{k,l\}\epsilon r} I_{kl}$ and I_{kl} still given by eq.(3). Here α_s refers to the α-coefficient of any one (chosen arbitrarily) of the M groups. This partitioning again leads to eq.(5) (with $\alpha \to \alpha_s$ on the right hand side) and therefore similar level-correlations in terms of the transition parameter $Y = 2\log\prod_r \frac{y_r}{|y_r - 1|}$ with $y_r = \frac{\alpha_r}{\alpha_s}$.

Finally let us consider the case of WRBM when mean value of the diagonal elements increases along the main diagonal: $< H_{nn} >= \gamma f(n)$. For simplification, we still take $\rho(H)$ to be a Gaussian: $\rho(H) = C\mathrm{e}^{-\sum_{k\leq l}\alpha_{kl}(H_{kl} - \gamma f(k)\delta_{kl})^2}$ with all distinct α_{kl}. Proceeding as before, one again obtains eq.(2) with I_{kl} given by eq.(3) but now, instead of applying the partitioning of the sum, we just evaluate $2\sum_{k\leq l}\frac{\partial P}{\partial \alpha_{kl}}\alpha_{kl}^2 g_{kl}$. This leads to a F-P equation

without the linear drift term,

$$\frac{\partial P(\{\mu_i\},Y)}{\partial Y} = \sum_n \frac{\partial}{\partial \mu_n}\left[\frac{\partial}{\partial \mu_n} + \sum_{m\neq n}\frac{\beta}{\mu_m-\mu_n}\right]P \tag{8}$$

where $Y = \sum_{k\leq l}(2\alpha_{kl}g_{kl})^{-1}$ and the steady state $P_\infty = \prod_{i<j}|\mu_i-\mu_j|^\beta$ is obtained in limit $Y\to\infty$ or $\alpha_{kl}\to 0$. Similarly for the degenerate case, when all the α_{kl} can be divided into M groups with distinct values $\alpha_r = \alpha_{kl}g_{kl}$, $r = 1\to M$, $2\sum_{r=1}^{M}\frac{\partial P}{\partial \alpha_r}\alpha_r^2$ can be used to obtain eq.(8) where now $Y = \sum_r(2\alpha_r^{-1})$. An application of the same transformation $\Psi = P/\sqrt{P_\infty}$ reduces eq.(8) in the "Schroedinger equation" form with Hamiltonian $\hat{H}$, $\hat{H} = \sum_i \frac{\partial^2}{\partial\mu_i^2} - \frac{1}{4}\sum_{i<j}\frac{\beta(\beta-2)}{(\mu_i-\mu_j)^2}$, which is same as CM Hamiltonian except for the absence of confining potential and, as shown in [20], has essentially the same solution for Ψ (without the Gaussian factor) and therefore $P(\mu,Y|H_0)$. However, for the evaluation of $P(\mu,Y)$, one needs to integrate over all the initial conditions through which the non-zero mean values of the diagonals enter in the calculation and may influence the statistics for large γ-values. Nontheless, in large N-limit, the hierarchic relations among the correlators are of the same type as in other RBM cases [19,20] with LDOS satisfying the eq.(6) [20]. Here the choice of H_0 from a Poisson ensemble ($\rho(H_0)\propto \mathrm{e}^{-\frac{\alpha}{2}\sum_j(H_{jj}-\gamma f(j))^2}$) gives the initial value of $Y=\alpha^{-1}$. With $Y\to\infty$ corresponding to a SG type ensemble, therefore, WRBM also appear as the intermediate ensembles in the Poisson $\to$ SG type transition.

In this paper, we have analytically studied the response of energy levels of complex quantum systems to the changing distribution of matrix elements of their hamiltonians. Our results indicate that the n^{th} order correlations at a given variance are the same as the corresponding eigenvalue-correlations of hamiltonian $H = H_0 + \tau V$ at a given τ value with H_0 taken from a Poisson ensemble and V taken from a SG ensembles. This analogy also extends to the 2^{nd} order parametric correlators however the Markovian nature of F-P equation restricts from making similar conclusions about higher orders. The intermediate ensembles arising in the VV transition correspond to various types of RMEs, and as our mapping suggests, the results for most of them can be obtained by studying just one transition in full detail, namely, an appropriate initial ensemble $\to$ SGE caused due to a SG perturbation; the

knowledge of their correlations can help us in statistical studies of many important physical properties of complex systems. Further, as discussed in [3], some of the RBMEs can also be connected to various other types of ensembles; a knowledge of the properties of former will therefore help in statistical analysis of the latter. It should also be possible to apply this technique to the ensembles of Unitary matrices (periodic RBMs) as well.

Our study still leaves many questions unanswered. At present, it is not clear whether our method can be extended to non-Gaussian distributions as well, at least in some limit. We are also unable to say whether a similar mapping can also be done for the eigenvector statistics. In this connection, however, it should be mentioned that analogy of equations governing the eigenvalue distribution leads to a similar form of equations for the distribution of matrix elements [2] and, therefore, it should somehow manifest itself in eigenvector distributions too.

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